# Implementing the Relaxation Damping of Air into the Navier-Stokes Equation

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## Nomenclature

$c$ = velocity of sound, m/s
$d$ = characteristic length, m
$f$ = frequency, Hz
$f_O, f_N$ = relaxation frequencies of oxygen and nitrogen, Hz
$h$ = molar concentration of water vapor, %
$G_{O,N}$ = damping constants of oxygen and nitrogen, s/m
$I$ = $3 \times 3$ identity tensor
$p$ = pressure, pressure of sound, $N/m^2$
$S$ = stress tensor, $N/m^2$
$St$ = Strouhal number
$T$ = temperature, K
$t$ = time, s
$t^*$ = integration time
$v$ = flow velocity in air, m/s
$W_{dil}$ = dilatations tensor, 1/s
$W_{dist}$ = distortion tensor, 1/s
$x, x_i$ = Cartesian coordinates, m, $i = 1, 2, 3$
$\alpha$ = sound attenuation, 1/m
$\eta$ = shear viscosity, kg/ms
$\lambda$ = vibrational relaxation viscosity, kg/ms
$\mu$ = rotational relaxation viscosity, kg/ms
$\rho$ = density of air, $kg/m^3$
$\tau$ = tension, $N/m^2$

*Subscripts*
N = nitrogen
O = oxygen
R = rotation
0 = standard atmosphere

## I. Introduction

REDUCING the Navier-Stokes equation to small flow velocities, the acoustic wave equation can be obtained. This equation contains an acoustic absorption which is proportional to the shear viscosity. Kirchhoff added a second absorption effect, due to the irreversible heat dissipation within the acoustic wave. These two terms describe the classical Stokes-Kirchhoff absorption in gases. But in 1933, when Knudsen[1] carried out experiments in air under atmospheric conditions, he discovered a much higher level of sound absorption. This absorption exceeded by a factor of about 20 the absorption predicted by Stokes and Kirchhoff. In the same year Kneser[2] explained the discrepancy by introducing a third damping phenomenon, that of relaxation. In particular, Kneser found that the vibrational relaxation of oxygen is responsible for enhanced acoustic attenuation in air and in the infrasonic range, the vibrational relaxation of nitrogen. Moreover, by using this theory it was possible to explain the sensitive influence of natural humidity on acoustic absorption in the air. Although the vapor is a small fraction in air it changes the acoustic absorption by orders of magnitude. The water vapor catalyzes the stimulation of the vibrational degrees of freedom. But not only the vibration, also the rotation of air molecules cause relaxation effects.

The task of this investigation is to implement the theoretically and experimentally proven relaxation damping into the Navier-Stokes equation for air.

## II. Sound Absorption

To obtain the acoustic absorption due to the shear viscosity $\eta$, the classical Navier-Stokes equation

$$\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) = -\mathrm{grad}\, p + \eta \Delta v + \frac{1}{3}\eta\, \mathrm{grad}\, \mathrm{div}\, v \tag{1}$$

is linearized. Using the linearized material and continuity equations, the acoustic equation including viscous shear effects is obtained

$$\frac{\partial^2 p}{\partial t^2} - c^2 \Delta p - \frac{4\eta}{3\rho}\frac{\Delta \partial p}{\partial t} = 0 \tag{2}$$

A plane sound wave propagating in the direction $x$ with frequency $f$

$$p = p_0 \exp(-\alpha_s x) \sin 2\pi f(t - x/c) \tag{3}$$

substituted in the equation for sound pressure, Eq. (2) yields the relation between the shear viscosity $\eta$ and the attenuation of sound $\alpha_s(\alpha_s \ll 2\pi f/c)$

$$\alpha_s = 8\pi^2 f^2 \eta / 3\rho c^3 \tag{4}$$

The shear mechanism causes only a small portion of the atmospheric sound absorption. There is also the Kirchhoff attenuation by heat dissipation, but the relaxation effects are dominant. First, the rotational relaxation of the air molecules gives the sound absorption[4,5]

$$\alpha_R = 2\pi^2 f^2 \mu / \rho c^3 \tag{5}$$

with $\mu$ as the rotational relaxation viscosity. According to the noise measurements by Greenspan[3] this viscosity in air is $\mu = 0{,}6\eta$. The main attenuation in the atmosphere up to aerodynamically relevant frequencies is caused by relaxation of the vibrational degrees of freedom. This relaxation process is characterized by the damping constants $G_O$ and $G_N$ and the relaxation frequencies $f_O$ and $f_N$ and gives rise to the acoustic attenuations $\alpha_N$ and $\alpha_O$ for the atmospheric nitrogen and oxygen[4,5]

$$\begin{aligned} \alpha_O &= G_O f^2/(f_O + f^2/f_O) \\ \alpha_N &= G_N f^2/(f_N + f^2/f_N) \end{aligned} \tag{6}$$

Numerous measurements on acoustic attenuation in air have been made. In the following, the International Standard Organization-norm[6] draft is applied. According to Ref. 6 the numeric values for the damping constants $G_O$ and $G_N$ and the relaxation frequencies $f_O$ and $f_N$ are

$$\begin{aligned} G_O &= 0.01275(T/T_0)^{-5/2} \exp(-2239.1/T) \\ G_N &= 0.1068(T/T_0)^{-5/2} \exp(-3352.0/T) \end{aligned} \tag{7}$$

$$\begin{aligned} f_O &= (p/p_0)\{24 + (4.04 \cdot 10^4)(h)[(0.02 + h)/(0.391 + h)]\} \\ f_N &= (p/p_0)(T/T_0)^{-1/2}[9 + 280h \exp\{-4.170[(T/T_0)^{-1/3} - 1]\}] \end{aligned} \tag{8}$$

where $p_0$ and $T_0$ are the reference values of the standard atmosphere

$$p_0 = 101.325 \text{ kPa } (= 1 \text{ atm}), \qquad T_0 = 293.15 \text{ K } (= 20°\text{C}) \tag{9}$$

The effects of temperature $T$ and molar concentration of water vapor $h$ on the relaxation frequencies $f_O$ and $f_N$ are presented in Table 1. The surprising effect is the strong catalytic influence of air humidity. A molar concentration of $h = 1\%$ (corresponding to a relative humidity of 43% at 20°C) increases the relaxation frequency $f_O$ of oxygen from 24 to 29,600 Hz.

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**Table 1 Vibrational relaxation frequencies $f_O$ and $f_N$ and vibrational viscosities $\lambda_O$ and $\lambda_N$ for oxygen and nitrogen in standard atmosphere[a]**

| Standard atmosphere ISO standard[6] | | | | Vibr. relaxation frequency | | Vibr. relaxation viscosity | |
|---|---|---|---|---|---|---|---|
| $H$, km | $T$, K | $P$, kPa | $h$, % | $f_O$, Hz | $f_N$, Hz | $\lambda_O$, kg/ms | $\lambda_N$, kg/ms |
| 0 | 288 | 101.3 | 1.009 | 29,978 | 287.09 | $4.593 \times 10^{-4}$ | $8.428 \times 10^{-3}$ |
| 2 | 275 | 79.5 | 0.606 | 12,079 | 132.94 | $6.793 \times 10^{-4}$ | $9.036 \times 10^{-3}$ |
| 4 | 262 | 61.6 | 0.358 | 4,452 | 60.77 | $1.051 \times 10^{-3}$ | $9.217 \times 10^{-3}$ |
| 6 | 249 | 47.2 | 0.098 | 456.2 | 15.53 | $5.566 \times 10^{-3}$ | $1.568 \times 10^{-2}$ |
| 8 | 236 | 35.6 | 0.016 | 28.52 | 4.808 | $4.566 \times 10^{-2}$ | $2.027 \times 10^{-2}$ |
| 10 | 223 | 26.4 | 0.006 | 10.39 | 3.026 | $5.974 \times 10^{-2}$ | $1.169 \times 10^{-2}$ |

[a]Accuracy better than 0.1%. Rotational relaxation viscosity in air is $\mu = 0.6\eta$ (Ref. 4, p. 553).

## III. Viscous Stress Tensors

The differential motion of a fluid with the velocity $v = v(x_i, t)$ is given by the velocity tensor $\nabla v = \{\partial v_i/\partial x_j\}$. The antisymmetric part of $\nabla v$ describes the flow rotation and needs not to be considered here. The symmetric tensor $\nabla v_{\text{sym}}$ consists of the distortion tensor $W_{\text{dist}}$ and the dilatation tensor $W_{\text{dil}}$. This corresponds mathematically to the separation of $\nabla v_{\text{sym}}$ in a tensor $W_{\text{dist}}$ with zero spur and in a spherically symmetric tensor $W_{\text{dil}}$ ($\nabla v^T$= transponded tensor, $I = 3 \times 3$ identity tensor)

$$(\nabla v)_{\text{sym}} = (\nabla v + \nabla v^T)/2 = W_{\text{dist}} + W_{\text{dil}} \tag{10}$$

$$W_{\text{dist}} = (\nabla v)_{\text{sym}} - (1/3)I \operatorname{div} v \tag{11}$$

$$W_{\text{dil}} = (1/3)I \operatorname{div} v \tag{12}$$

According to the Stokes hypothesis that the viscous stresses $\tau_{ij}$ are solely caused by the distortion tensor $W_{\text{dist}}$, representing the constant volume behavior, the stress tensor $S_s$ is defined by (the factor 2 was introduced by Stokes to obtain Newton's shear stress law)

$$S_s = 2\eta W_{\text{dist}} \tag{13}$$

The total volume change in the fluid is given by the dilatation tensor $W_{\text{dil}}$. As the relaxation damping is proportional to the change of temperature, and therefore to the volume change, the stress tensor $S_R$ describing rotational relaxation gives

$$S_R = 3\mu W_{\text{dil}} = \mu I \operatorname{div} v \tag{14}$$

With the same argumentation it is postulated that the vibrational relaxation stresses are also proportional to $W_{\text{dil}}$. The formal proportional factors $\lambda_0$ and $\lambda_N$ and can be identified later as the vibrational relaxation viscosities of the oxygen and nitrogen. Additionally, the frequency dependance (6) of vibration relaxation is to be considered. The transformation from the frequency domain into the time domain is done by convolution. Especially, the frequency behavior according to Eq. (6) corresponds to

$$f^2/(f^2 + f^2_{\text{O,N}}) \to \exp(-2\pi f_{\text{O,N}} t^*) \tag{15}$$

and gives the stress tensors $S_O$ and $S_N$ of the vibrational relaxation for the atmospheric oxygen and nitrogen

$$S_O = \lambda_O I \int_{t^*=0}^{\infty} \exp(-2\pi f_O t^*) \operatorname{div} v(t - t^*) \, d2\pi f_O t^*$$

$$S_N = \lambda_N I \int_{t^*=0}^{\infty} \exp(-2\pi f_N t^*) \operatorname{div} v(t - t^*) \, d2\pi f_N t^* \tag{16}$$

The variable for integration $t^*$ has the dimensions of time and represents the retarded time, starting at the present time $t$. Physically, the relaxation tensor depends on the present velocity $v(t)$, as well as the past $v(t - t^*)$.

## IV. Flow Equation for Air

The flow equation for air with the viscous stress tensors $S_S$, $S_R$, $S_O$, and $S_N$ is given by

$$\frac{\rho \partial v}{\partial t} + \rho v \cdot \nabla v = -\operatorname{grad} p + \operatorname{div} S_S + \operatorname{div} S_R + \operatorname{div} S_O + \operatorname{div} S_N \tag{17}$$

Assuming constant shear and relaxation viscosities $\eta$, $\mu$, $\lambda_O$, and $\lambda_N$, the formal calculation gives the flow equation for the atmospheric air

$$\begin{aligned} \frac{\rho \partial v}{\partial t} &+ \rho v \cdot \nabla v = -\operatorname{grad} p \\ &+ \eta \left( \Delta v + \frac{1}{3} \operatorname{grad} \operatorname{div} v \right) + \mu \operatorname{grad} \operatorname{div} v \\ &+ \lambda_O \int_{t^*=0}^{\infty} \exp(-2\pi f_O t^*) \operatorname{grad} \operatorname{div} v(t - t^*) \, d2\pi f_O t^* \\ &+ \lambda_N \int_{t^*=0}^{\infty} \exp(-2\pi f_N t^*) \operatorname{grad} \operatorname{div} v(t - t^*) \, d2\pi f_N t^* \end{aligned} \tag{18}$$

The first line corresponds to the Navier-Stokes equation; the additional terms reflect viscous relaxation forces for atmospheric air. The convolution integrals appear complicated, but they can be evaluated numerically. The calculation process gives knowledge about the past $v(t - t^*)$, which needs not be calculated additionally. Also the integral converges very fast due to the exponential convolution function.

To determine the unknown relaxation viscosities $\lambda_O$ and $\lambda_N$, and also to validate, the flow equation (18) will be transformed by linearization, and using the material and the continuity conditions in the equation for the sound pressure ( $\dot{p} = \partial p/\partial t$)

$$\begin{aligned} \ddot{p}/c^2 &= \Delta p + (4\eta/3\rho c^2 + \mu/\rho c^2)\Delta \dot{p} \\ &+ \left(\frac{\lambda_O}{\rho c^2}\right) \int_{t^*=0}^{\infty} \exp(-2\pi f_O t^*) \Delta \dot{p}(t - t^*) \, d2\pi f_O t^* \\ &+ \left(\frac{\lambda_N}{\rho c^2}\right) \int_{t^*=0}^{\infty} \exp(-2\pi f_N t^*) \Delta \dot{p}(t - t^*) \, d2\pi f_N t^* \end{aligned} \tag{19}$$

A plane acoustic wave in the $x$ direction with the frequency $f$

$$p = p_0 \exp(-\alpha_S x - \alpha_R x - \alpha_O x - \alpha_N x) \sin 2\pi f(t - x/c) \tag{20}$$

inserted in Eq. (19) one gets the sound attenuations $\alpha_S$ by shear, $\alpha_R$ by rotational, and $\alpha_O$ and $\alpha_N$ by vibrational relaxation

$$\alpha_S = \frac{8\pi^2 \eta f^2}{3\rho c^3}, \qquad \alpha_R = \frac{2\pi^2 \mu f^2}{\rho c^3} \tag{21}$$

$$\alpha_O = \frac{2\pi^2\lambda_O f_O f^2}{\rho c^3(f_O + f^2/f_O)}, \qquad \alpha_N = \frac{2\pi^2\lambda_N f_N f^2}{\rho c^3(f_N + f^2/f_N)} \tag{22}$$

Comparing Eq. (22) with Eq. (6) one gets the relaxation viscosities of oxygen and nitrogen $\lambda_O$ and $\lambda_N$ in the air.

$$\lambda_O = \frac{\rho c^3 G_O}{2\pi^2 f_O}, \qquad \lambda_N = \frac{\rho c^3 G_N}{2\pi^2 f_N} \tag{23}$$

In Table 1 the numerical values $\lambda_O$ and $\lambda_N$ of the standard atmosphere calculated with Eqs. (7), (8), and (23) are given.

## V. Approximated Flow Equation in Air

As an alternative to the exact solution of Eq. (18), the convolution integrals may be substituted by Taylor series expansions with

$$v(t-t^*) = \sum_{n=0} \frac{1}{n!}\frac{\mathrm{d}^n v}{\mathrm{d}t^n}(-t^*)^n = v - \left(\frac{\partial v}{\partial t} + v\nabla v\right)t^* \pm \quad \ldots \tag{24}$$

The flow equation in air is as follows:

$$\begin{aligned}\frac{\rho\partial v}{\partial t} + \rho v \cdot \nabla v &= -\mathrm{grad}\, p + \eta\left(\nabla v + \frac{1}{3}\mathrm{grad\ div}\ v\right)\\ &+ \mu\ \mathrm{grad\ div}\ v + \lambda_O\ \mathrm{grad\ div} \sum_{n=0}\left(\frac{-1}{2\pi f_O}\right)^n \frac{\mathrm{d}^n v}{\mathrm{d}t^n}\\ &+ \lambda_N\ \mathrm{grad\ div} \sum_{n=0}\left(\frac{-1}{2\pi f_N}\right)^n \frac{\mathrm{d}^n v}{\mathrm{d}t^n}\end{aligned} \tag{25}$$

For air flow under environmental conditions, the zeroth-order term $n = 0$ may be sufficient. For normal air humidity values, the relaxation frequencies are well separated; for nitrogen, $f_N$ = 100–300 Hz, whereas for oxygen, $f_O$ = 20,000–30,000 Hz. Defining the Strouhal numbers $St_N$ and $St_O$ with $d_c$ the characteristic length and $v_c$ the characteristic velocity

$$St_N = f_N d_c / v_c, \qquad St_O = f_O d_c / v_c \tag{26}$$

For flow velocities with a Strouhal number $St = fd_c/v_c$ the flow equation (25) can be approximated by

$$\begin{aligned}\frac{\rho\partial v}{\partial t} + \rho v \cdot \nabla v &= -\mathrm{grad}\, p + \eta\Delta v\\ &+ \left\{\begin{matrix}(1/3)\eta + \mu + \lambda_O + \lambda_N\\ (1/3)\eta + \mu + \lambda_O\\ (1/3)\eta + \mu\end{matrix}\right\} \mathrm{grad\ div}\ v\\ \text{for} &\left\{\begin{matrix}St < St_N\\ St_N < St < St_O\\ St > St_O\end{matrix}\right\}\end{aligned} \tag{27}$$

In the standard atmosphere at sea level the shear viscosity is $\eta = 1.7 \times 10^{-5}$ kg/ms, and the relaxation viscosity of atmospheric oxygen $\lambda_O = 4.6 \times 10^{-4}$ kg/ms. In this case the term grad div $v$ is 79.2 times higher than in the classical Navier-Stokes equation without the vibrational relaxation effect.

## References

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# Time-Of-Flight Mass Spectrometer for Impulse Facilities

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## Introduction

At present, impulse facilities such as shock tunnels and expansion tubes are the only practical ground-based means of simulating aerodynamic conditions at speeds in excess of 3.5 km/s. The duration of test flow, however, is of the order of a millisecond, requiring specialized instrumentation. It is especially desirable to measure species concentrations in the test flow since the chemical state of the test gas may be altered by the high transient temperatures produced. Also, the flow of test gas in a reflected shock tunnel is followed by a flow of driver gas, usually of a different composition. The breakdown of the interface between these two gas slugs causing mixing of the driver and test gases typically limits the available test time. Measurements of species concentrations are the most sensitive methods of determining the duration of test flow and have been used in the past.[1] Furthermore, studies of supersonic combustion ramjets would benefit from measurements of species concentrations downstream of fuel injection.

A quadrupole mass spectrometer was used by Crane and Stalker[2] in a reflected shock tunnel. They used a two-stage hollow conical skimmer gas sampling system. The short duration of the flow meant that the ions from only a single species of molecule could be recorded during each test flow. To obtain relative concentrations, Crane and Stalker relied upon sampling multiple test flows and measured the total ion production of each shot for normalization. This method assumed repeatability of the test flow and of the behavior of the mass spectrometer despite variations in total ion production. They estimated errors in their relative species concentration measurements of 30%.

## Experimental Design

The instrument reported here was designed for use in T4, an existing free piston reflected shock tunnel.[3] At the upstream end are three coaxial, hollow conical skimmers used to sample the flow (Fig. 1). Behind the second and third skimmers are vacuum chambers evacuated by diffusion pumps. There is no vacuum chamber behind the first skimmer as the flow through it vents back out into the freestream flow. Gas passing through the skimmers is expanded and collimated into a molecular beam entering the interior high vacuum chamber. At 20 mm behind the final skimmer, an electron beam intersects the molecular beam, causing ionization of some molecules. The ions produced are accelerated along a meter-long flight tube to an electron multiplier particle detector. The signal from this particle detector is recorded on a digital storage oscilloscope.

An essential aspect of the success of the instrument is that the production of the molecular beam be achieved as quickly as possi-

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